

# A Kirchhoff-like conservation law in Regge calculus

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## Abstract.

Simplicial lattices provide an elegant framework for discrete spacetimes. The inherent orthogonality between a simplicial lattice and its circumcentric dual yields an austere representation of spacetime which provides a conceptually simple form of Einstein's geometric theory of gravitation. A sufficient understanding of simplicial spacetimes has been demonstrated in the literature for spacetimes devoid of all non-gravitational sources. However, this understanding has not been adequately extended to non-vacuum spacetime models. Consequently, a deep understanding of the diffeomorphic structure of the discrete theory is lacking. Conservation laws and symmetry properties are attractive starting points for coupling matter with the lattice. We present a simplicial form of the contracted Bianchi identity which is based on the E. Cartan moment of rotation operator. This identity manifest itself in the conceptually-simple form of a Kirchhoff-like conservation law. This conservation law enables one to extend Regge Calculus to non-vacuum spacetimes and provides a deeper understanding of the simplicial diffeomorphism group.

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## 1. Introduction

Regge Calculus (RC) provides a natural framework for the description of discrete spacetimes [1, 2]. It has been applied to problems ranging from quantum gravity to classical relativity [3]. In recent years RC has been used in quantum gravity to construct emergent spacetime models [4] and to investigate the spin-foam approach and its perturbative regime [5, 6, 7, 8]. In particular, [4] utilizes a pre-geometric framework wherein the spacetime is an emergent structure from an underlying quantum system. The unitary operators of the discrete quantum system act as source terms for the ensuing spacetime. If an approach of this kind is to utilize RC as the coupling to the underlying quantum dynamics, it is incumbent upon us to understand the nature of the coupling of matter and fields to a RC lattice spacetime.

Most early attempts to include matter and fields in RC concentrated on spacetimes with a high degree of symmetry [9, 11] or in specific cases [10, 12]. A more general approach was given by [13]; however, this approach introduced a simplex-based action which does not appear to be natural to us given the vertex-based description indicated by the contracted Bianchi identity. A general theory of the coupling of source terms to RC has not been fully developed. It is the goal of this manuscript to provide the first steps to construct a simplicial representation for the stress-energy of the non-gravitational sources. To do so we first examine the extent to which a simplicial lattice can guarantee the conservation of energy-momentum. We then develop more fully the contracted Bianchi identity on the lattice. This identity, when augmented by the Einstein equations, yields the conservation of energy and momentum.

We know no shorter route to derive the contracted Bianchi identity than by using the topological tautology that the boundary of a boundary is zero, i.e. the boundary-of-a-boundary principle (BBP). The BBP appears twice over in each of nature's four fundamental interactions, once in its 1-2-3-dimensional form, and once in its 2-3-4-dimensional form [14, 2].

The BBP has been used in RC to obtain a discrete version of the contracted Bianchi identity [15, 16, 17, 18]. However, the interpretation of this conservation law has been a source of some debate particularly over the exactness of the identity. While the topological principle itself is exact and thoroughly studied in RC [18], the transition from a continuum to a discrete spacetime forces one to apply these topological identity to non-infinitesimal rotations. Unlike the infinitesimal rotation operators in the continuum, finite rotations do not ordinarily commute. The transition from the continuum to the discrete case must be handled with care. We emphasize that the derivation presented here will not ordinarily produce an exact identity due to the non-commuting nature

of finite rotations. Nevertheless what one loses in exactness one gains in simplicity. In particular, the integrated Einstein tensor is doubly projected along its edge and this allows one to write down the contracted Bianchi identity as a Kirchhoff-like conservation principle. These identity are second-order convergent [15] and valid locally at any event in a spacetime.

The contracted Bianchi identity for RC has clear implications for the coupling of energy-momentum to the lattice as well as to our understanding of diffeomorphism invariance in RC. Furthermore, if we expect the quantization of spacetime to produce an inherently discrete spacetime, then grasping the meaning of the BBP in a discrete theory becomes essential to understanding the quantum theory of gravity. RC serves naturally as an underlying framework since simplicial spacetimes provide one of the most elegant and universal descriptions of discrete spacetime [19].

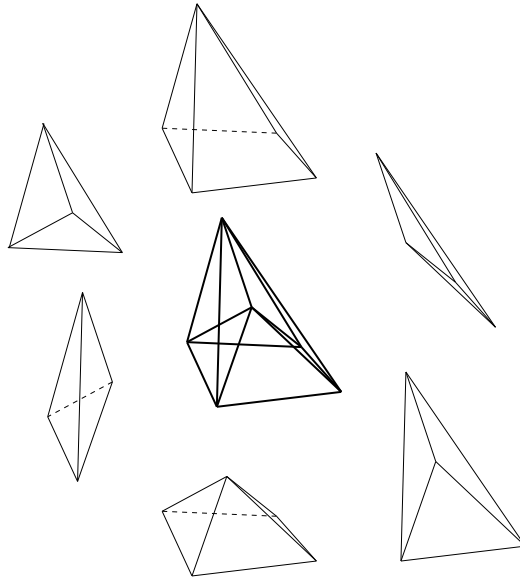
In Sec. 2 we review the BBP and its role in the fundamental forces of nature. The importance of this identity stems from its purely topological foundation. The Cartan construction of the moment of rotation trivector in RC is reviewed before applying the BBP directly to the simplicial lattice in Sec. 3. We conclude in Sec. 4 with our future plans to couple a generic stress-energy tensor to the geometric content of the Regge lattice.

## 2. Boundary of a Boundary Principle: The Guiding Topological Principle

In any fundamental field theory (electrodynamics, Yang – Mills, general relativity) the conservation of source is introduced in such a way that it is satisfied for any field. This is equivalent to saying that it does not impose any restrictions on the field itself but rather puts constraints on the source of the field (charge in electrodynamics, energy-momentum in general relativity). This feature is conditioned only by the requirement that the field is described as the curvature of a connection on the appropriate vector bundle that is responsible for the correct implementation of the field symmetries [20].

The universality of this feature follows from the fact that it is induced by (but not totally reduced to) the simple topological identity that the boundary-of-a-boundary is equal to zero [21]. Application of this principle to spacetime is achieved by associating to it a chain complex (say by simplicial or any other triangulation) with the standard boundary operator based on the rules of orientations of the boundaries. As an example, we can examine a discrete representation spacetime wherein the spacetime geometry is tiled by 4-dimensional polytopes. The geometry interior to each of these infinitesimal polytopes is irrelevant and, for pictorial representations, can be thought of as flat Minkowski geometry. Let us examine one of these polytopes,  $V^*$ , which is the local

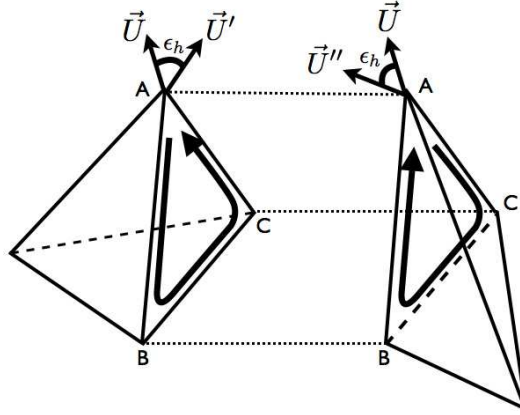
neighborhood of an event,  $V$ . This polytope is bounded by 3-dimensional polyhedra (Figure 1). Any two adjacent polyhedra on the boundary of  $V^*$  share a common 2-dimensional face. In other words, in this 4-dimensional region of spacetime no 2-dimensional polygonal faces are exposed. In general relativity, any flow of stress-energy (or equivalently the dual of the Cartan moment of rotation) into one of the 3-dimensional bounding polyhedra is exactly compensated by an equal flow of stress-energy (Cartan moment of rotation) out of an adjacent polyhedron. This guarantees conservation of source in  $V^*$ .



**Figure 1.** *The polyhedral boundary of a 4-polytope:* This illustration shows the 2-dimensional projection of a typical 4-dimensional polytope,  $V^*$ , of the circumcentric dual (Voronoi) spacetime. It is dual to a vertex,  $V$ , of the simplicial (Delaunay) spacetime. This 4-polytope is bounded by six polygons (shown exploded off into the perimeter of the polytope). These 4-polytopes are ordinarily not simplexes nor are their bounding polyhedra. The orientation of  $V^*$  induces an orientation on each of its polyhedral faces,  $L^*$ . The orientation of each polyhedron consequently induces an orientation on each its polygonal faces. However, each 2-face is shared by two polyhedra thereby inducing equal and opposite orientations on it. In this sense none of these polygonal faces are exposed and their orientations cancel. This is the origin of the BBP principle in its 2-3-4 dimensional form.

For each  $V^*$ , one would like to sum over each of its unexposed 2-dimensional boundaries – the meeting place of two of the polyhedral boundaries of  $V^*$  (Figure 2). These polyhedral boundaries induce opposing orientations on each of the 2-dimensional faces. Therefore when one sums over all of the 2-boundaries of all the 3-boundaries,

two contributions are found for each polygon each of equal magnitude but opposite orientations. These identically cancel one another leaving the boundary-of-a-boundary identically equal to zero.



**Figure 2.** *BBP as a Geometric Identity:* Here two adjoining faces of the 3-dimensional boundary of a 4-dimensional volume are depicted with their induced orientation. The orientation of the 2-dimensional area is seen to be opposite for the adjoining 3-volumes such that in the sum over the boundary of the boundary these areas cancel one another. Furthermore, if a vector  $\vec{U}$  is parallel transported around the area adjoining the two 3-volumes, then the vector will ordinarily come back rotated. When the area is associated with the left 3-volume, the vector  $\vec{U}$  comes back rotated as  $\vec{U}'$ , but when the area is associated with the right 3-volume it will come back as  $\vec{U}''$ . The rotation in both cases is in the same plane and rotated by the same amount but in opposite directions of rotation.

In applications to continuum field theories the boundary of a boundary relation of the chain complex is translated into the relation co-boundary of a co-boundary of the dual de Rham co-chain complex of differential forms. The exterior derivative acts as the co-boundary operator [21]. This duality is established by adding rotations caused by parallel translations of vectors around the 2-faces (or Cartan moments of these rotations) of an infinitesimal 3-simplex (or 4-simplex for moments of rotations). These rotations are expressed as products of Riemannian curvature tensors on each face and the oriented element of area associated with the face. This operation, when applied to finite structures, is ambiguous and cannot be performed in a consistent way. The ambiguity arises for two reasons: finite rotations do not commute, and tensor quantities are being computed at different points (on different faces) and then added. These difficulties disappear in the infinitesimal limit.

In particular the application of 1-2-3 dimensional BBP in general relativity reduces

to computing the sum of rotations caused by parallel transport around all 2-faces of a 3-simplex. When expanded in Taylor series with respect to displacements along the edges [2] it produces terms of second and the third order (the higher orders are of no interest in this computation). The ambiguity caused by parallel transporting tensor quantities to a common point (necessary for addition) introduces errors of the fourth and higher order. These errors can be neglected. The requirement that the second order term vanishes leads to the conclusion that the Riemannian curvatures on the faces of the 3-simplex are not linearly independent, while the requirement that the third order term vanishes implies the ordinary Bianchi identity.

Likewise, application of 2-3-4 dimensional BBP amounts to adding Cartan moments of rotation over all 3-faces of a 4-simplex (or polytope). The Taylor series expansion proceeds as before, with terms up to the third order disappearing because of relations imposed by the 1-2-3 dimensional form of the BBP. The errors generated by the ambiguity of parallel translation are now of the fifth and higher order. The contracted Bianchi identity arises from the fourth order term of the expansion.

### 3. Discrete Bianchi Identity

Regge Calculus is based on a lattice of flat 4-dimensional simplexes that form a curved piece-wise flat manifold. The curvature is concentrated as conical singularities on each of the co-dimension 2 triangular hinges. To define the curvature we need to associate an “area-of-circumnavigation” to each triangular hinge. This provides a finite area over which we can distribute the curvature. Fortunately, the circumcentric dual lattice has been shown to arise naturally in RC and provides an appropriate area [22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32]. Correspondingly, it has been shown that the circumcentric 3-volumes naturally define the moment-of-rotation operators and discrete RC equations [15]. Finally, we postulate here that the circumcentric 4-volume ( $V^*$ ) dual to a vertex ( $V$ ) defines a natural domain to apply the Cartan BBP in its 2-3-4-dimensional form. Consequently the BBP in RC becomes the “co-boundary of the co-boundary” principle, although the geometric underpinnings are exactly the same.

In this section we derive the discrete form of the contracted Bianchi identity. We begin by emphasizing the central role that the Cartan moment-of-rotation trivector and the circumcentric dual lattice play in this derivation. In particular, we begin by re-expressing the familiar Regge equations in the Cartan prescription. This leads naturally to a Kirchhoff-like identity at each vertex inherently linked with the topological boundary-of-a-boundary identity. We conclude the section with an analysis of the convergence properties of the identity with the typical lattice edge length  $L$ .

### 3.1. The Regge-Einstein tensor and the Cartan moment of rotation

To begin the derivation we follow E. Cartan and examine the moment of rotation trivector. The dual to this trivector generates the Regge equation as well as the RC version of the Einstein tensor. Recall that there is one Regge equation for each edge,  $L$ , in the simplicial spacetime geometry,

$$\left( \begin{array}{c} \text{Regge equation} \\ \text{associated with} \\ \text{edge } L \end{array} \right) = \sum_{h \supset L} \frac{1}{2} L \cot(\theta_{Lh}) \epsilon_h, \quad (1)$$

where the sum is taken over each triangular hinge  $h$  sharing edge  $L$ ,  $\theta_{Lh}$  is the interior angle of  $h$  opposite  $L$ , and  $\epsilon_h$  is the deficit angle of hinge  $h$ .

The Cartan moment-of-rotation trivector is defined through a moment arm ( $dP$ ) reaching from a fulcrum to a rotation bivector. Each triangular hinge  $h$  in the simplicial spacetime has an associated rotation bivector ( $\mathcal{R}_h$ ) located at the circumcenter  $C$  of the hinge  $h$ . The orientation of  $\mathcal{R}_h$  is in a 2-plane,  $h^*$ , perpendicular to the hinge,  $h$ . The bivector is formed by two unit vectors separated by the usual RC deficit angle,  $\epsilon_h$ .

It is convenient to locate the fulcrum at one of the two endpoints edge  $L$ . We denote this fulcrum vertex as  $V$ , and by construction it is one of the vertices of hinge  $h$ . This freedom of choice is guaranteed by the ordinary Bianchi identity as we show below. This is in contrast to previous derivations of the Regge equations using the Cartan approach where the fulcrum was taken halfway along edge  $L$  [15, 31, 32].

With the fulcrum at  $V$  we can decompose the moment arm associated with hinge  $h$  into two vectors (Fig. 3),

$$(\text{Moment Arm})_{Lh} = P_L + dP_{Lh} \quad (2)$$

where  $P_L = \frac{1}{2}\mathbf{L}$  is the vector from the fulcrum  $V$  to point  $O$ , located at the center of edge  $L$ . This is also the center of three-dimensional circumcentric polyhedron  $L^*$ , defined to be dual to edge  $L$ . The other component of the moment arm,  $dP_{Lh}$ , is the vector from  $O$  to the circumcenter  $C$  of the hinge. This gives us two vectorial contributions to the moment arm; one ( $P_L$ ) is common to all 2-dimensional faces  $h^*$  of the dual polyhedron  $L^*$ , and another ( $dP_{Lh}$ ) is distinct for each of these 2-dimensional faces. The contribution common to all faces of  $L^*$  can be factored out of the sum of moments of rotations, so that

$$\sum_{h \supset L} (P_L + dP_{Lh}) \wedge \mathcal{R}_h = \underbrace{P_L}_{\text{common}} \wedge \sum_{h \supset L} \mathcal{R}_h + \sum_{h \supset L} dP_{Lh} \wedge \mathcal{R}_h. \quad (3)$$

The resulting sum over all rotations around  $L^*$  is simply the ordinary Bianchi identity for RC [1, 15],

$$\sum_{h \supset L} \mathcal{R}_h = \mathcal{O}(L^2). \quad (4)$$

In Eqs. 3-4 the sum over the hinges,  $h_L$ , sharing edge  $L$  could have equally been taken over the bounding polygons,  $h^*$ , of the dual polygon,  $V_L^*$ . There is a one-to-one correspondence between the  $h$  and  $h^*$ .

Using this approximate Bianchi identity we are justified in removing the common contribution to the moment arm in our sum over the moments-of-rotation. We see that the ordinary Bianchi identity allows us to freely choose the position of the fulcrum. A natural choice for the fulcrum is the vertex  $V$  and we use  $V^*$  to denote the dual 4-polytope to this vertex. Then each edge,  $L$ , emanating from vertex,  $V$ , has the moment arm,  $P_L = \frac{1}{2}\mathbf{L}$ , which is directed along edge,  $L$ . Since each edge  $L$  is dual to a corresponding 3-polytope,  $L^*$ , the effective moment arm is

$$\left( \begin{array}{c} \text{Effective} \\ \text{Moment Arm} \end{array} \right) = dP_{Lh} = \frac{1}{2} L \cot \theta_{Lh} \hat{\mathbf{n}}, \quad (5)$$

which is the segment from the center of the edge ( $O$ ) to the circumcenter ( $C$ ) of the hinge (Figure 3).

We are now in a position to explicitly reconstruct the Regge equation associated to an edge  $L$ , and to construct the corresponding Regge-Einstein tensor. To define the moment of rotation trivector associated to hinge  $h$  and edge  $L$  we needed both the moment arm and the rotation bivector. Parallel transport of a unit vector around the 2-dimensional face  $h^*$  dual to the hinge  $h$  returns a unit vector rotated by an amount equal to the deficit angle ( $\epsilon_h$ ) associated with the hinge. Furthermore, the rotation bivector lies in the plane  $h^*$ , perpendicular to the hinge.

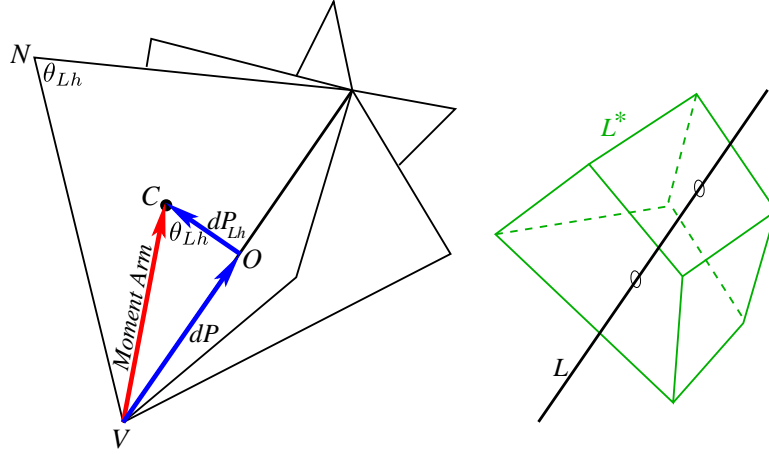
The dual of the Einstein tensor is expressible in terms of the moment of rotation trivector [2],

$$\int_{V^*} {}^* \mathbf{G} = \int_{\partial V^*} \star (\mathbf{dP} \wedge \mathcal{R}) = \mathbf{0}, \quad (6)$$

where the Hodge dual only acts in the space of values, i.e. on the moment of rotation trivector. In RC the moment of rotation trivector consists entirely of the parallelepiped formed by the moment arm  $dP_{Lh}$  and the two vectors defining the rotation bivector,

$$\left( \begin{array}{c} \text{Bivector} \\ \text{dual to} \\ \text{hinge } h \end{array} \right) = \mathcal{R}_h = \frac{\mathbf{L} \wedge V\vec{N}}{2A_h} \epsilon_h, \quad (7)$$





**Figure 3.** *Hinges and Moment Arm:* In the simplicial lattice each edge is common to multiple hinges  $h$  (left). The circumcentric 3-volume  $L^*$  dual to edge  $L$  has 2-dimensional boundaries dual to each of the hinges  $h$  (right). The parallel transport of a vector around the perimeter of these dual areas will result in a net rotation by an angle equal to the deficit angle,  $\epsilon_h$ , associated with the hinge,  $h$ . The moment of rotation is given by a moment arm  $P_L + dP_{Lh}$  wedge the rotation associated with the parallel transport around the dual area. However, the first term does not contribute as it is equal to zero by the ordinary Bianchi identity. On a given hinge, the effective moment arm is the vector from the edge to the center of rotation, i.e. the circumcenter of the hinge  $C$ , which has length  $(1/2)L \cot \theta_{Lh}$ .

which lies in the plane orthogonal to the triangular hinge  $h$ . This hinge is defined by the vectors  $\mathbf{L}$  and  $V\vec{N}$ , and has area  $A_h$ . The star dual of the parallelepiped returns a vector of length  $\epsilon_h$  and parallel to edge  $\mathbf{L}$ .

We can now construct the moment-of-rotation trivector. The dual moment of rotation associated with a hinge  $h$  containing the edge  $L$  is

$$\left( \begin{array}{c} \text{Dual Moment} \\ \text{of Rotation} \\ \text{for hinge } h \end{array} \right)_L = \star(dP_{Lh} \wedge R_h) \longrightarrow \underbrace{\frac{1}{2} \mathbf{L} \cot \theta_{Lh}}_{\text{Moment Arm}} \underbrace{\epsilon_h}_{\text{Rot'n}}. \quad (8)$$

The *total* dual moment of rotation over the Voronoi 3-volume  $L^*$  is then found by adding contributions from all hinges which share the edge  $L$ ,

$$\int_{\partial \mathcal{V}^*} \star(dP_{Lh} \wedge R_h) \longrightarrow \frac{1}{2} \sum_{h \supset L} \mathbf{L} \cot \theta_{Lh} \epsilon_h. \quad (9)$$

In the Cartan description of Einstein's theory [2, 14, 21] the Einstein tensor associated with a three dimensional region is the dual of the total moment of rotation tri-vector per unit three-volume. The two components of the Einstein tensor describe

the orientation of the three volume, and the orientation of the moment of the rotation tri-vector. In Regge calculus there is one equation per edge  $L$ , as can be seen when the Cartan moment of rotation is calculated over the Voronoi three-volume  $L^*$  [15]. The orthogonality between the simplicial (Delaunay) lattice and its circumcentric dual (Voronoi lattice) yields an Einstein tensor which is doubly projected along edge  $L$ . That is,

$$\left( \begin{array}{c} \text{Integrated Einstein} \\ \text{Tensor associated} \\ \text{with edge } L \end{array} \right) = \int_{V^*} {}^*G \longrightarrow \frac{G_{LL} L^*}{L} \mathbf{L}, \quad (10)$$

which is directed along edge  $L$  and has magnitude  $G_{LL} L^*$ .

Combining equations (6), (9) and (10) establishes the relationship between the Regge equations and the *integrated* simplicial Einstein equations [15],

$$G_{LL} L^* = \frac{1}{2} \sum_{h \supset L} L \cot \theta_{Lh} \epsilon_h. \quad (11)$$

This effectively defines the simplicial Einstein tensor  $G_{LL}$  at edge  $L$ .

Finally, we note that the simplicial Einstein tensor along the edge  $L$ , constructed using the sum of moments of rotations for the dual 3-volume  $L^*$ , is simply the geometric portion of the familiar Regge equation,

$$\frac{1}{2} \sum_{h \supset L} L \cot \theta_{Lh} \epsilon_h. \quad (12)$$

### 3.2. The contracted Bianchi identity

As shown above, the Regge equation and corresponding simplicial Einstein (or Regge-Einstein) tensor is naturally defined relative to the dual polyhedron  $L^*$ . It is therefore natural to define the 4-dimensional polytope  $V^*$  dual to the vertex  $V$  as the domain upon which we apply the Cartan BBP in its 2-3-4 dimensional form. To demand no net creation of source ( $\nabla \cdot G = \nabla \cdot T = 0$ ) in this spacetime region is to embody the essence of the contracted Bianchi identity.

As in the continuum (Sec. 2), we can provide a finite sum (“integral”) representation of the contracted Bianchi identity associated to the dual polytope  $V^*$  by summing over its polyhedral 3-boundaries  $L^*$ . By construction, each polyhedral 3-boundary of the dual polytope  $L^*$  is dual to one of the edges  $L$  of the simplicial lattice emanating from vertex  $V$ . This defines the domain of integration for the BBP. This completes two steps toward the BBP in RC by defining both the domain and the integrand (Sec. 3.1) for the BBP in RC.

The final step in deriving an expression for the conservation of moment of rotation in RC is achieved by summing over the dual 3-volumes  $L^*$  that bound the 4-volume  $V^*$ , which is dual to vertex  $V$ . However, care must be taken in evaluating this sum. Despite our choice of a common fulcrum, we have still decomposed the total moment arm into two vectors (one strictly in the tangent space of the vertex  $V$ , and one at the center of edge  $L$ ). Yet, RC provides a simple solution to what could be problematic. Both of these vectors lie in the tangent space of their associated hinge  $h$ . As such, the decomposition of the total moment arm becomes the standard decomposition of a vector in flat Minkowski spacetime. Summation over terms at a common point can be achieved in two equivalent, but separate approaches; (1) by parallel transporting the effective moment arm prior to inclusion in the moment of rotation trivector, or (2) parallel transporting the net moment of rotation trivector. We will consider the second approach.

In RC the integrated Einstein tensor (11) is not only evaluated along the edge  $L$ , it is also directed along  $\mathbf{L}$ . Since each simplicial edge  $L$  is by definition a geodesic in the lattice, any vector parallel transported along  $\mathbf{L}$  will maintain a constant angle with respect to  $\mathbf{L}$ . We take advantage of this property and individually transport each of the RC moment-of-rotation trivectors from the center of their respective edges to the vertex  $V$  which is common to all of these edges. We are then free to sum these moment-of-rotation vectors at  $V$ . Repeating this procedure across the lattice yields a 4-vector identity at each and every vertex  $V$ ,

$$\begin{pmatrix} \text{Net Moment} \\ \text{of Rotation} \\ \text{at vertex } V \end{pmatrix} = \sum_{L \supset V} \sum_{h \supset L} \frac{1}{2} \mathbf{L} \cot(\theta_{Lh}) \epsilon_h. \quad (13)$$

This is the simplicial form of the net moment of rotation at vertex  $V$  and must vanish by the 2-3-4 dimensional form of the BBP. However, as we have mentioned, the finite rotation operators do not ordinarily commute. This is important because we must apply our rotations in a given order. Nevertheless, the non-commutativity of the rotation operators can be made as small as one wishes by suitably refining the lengths of the simplicial lattice. Here, suitable refinement of the lattice is taken in the sense described in [33] where constant curvature barycentric subdivision is employed to refine the edge lengths by introducing new simplicial blocks and distributing curvature over the new subdivision of the simplexes. Under such refinements, the commutators for rotations scale as the deficit angles squared. Moreover, the deficit angles scale as the edge length squared as can be seen via their relation to the curvature  $(\text{Curvature}) = K = \epsilon_h/A_h^*$ . Consequently, the deficit angles scale as  $\mathcal{O}(L^2)$  and the commutators for rotations scale as  $\mathcal{O}(L^4)$ . This second order convergence is the origin of the approximation being

implemented. Therefore,

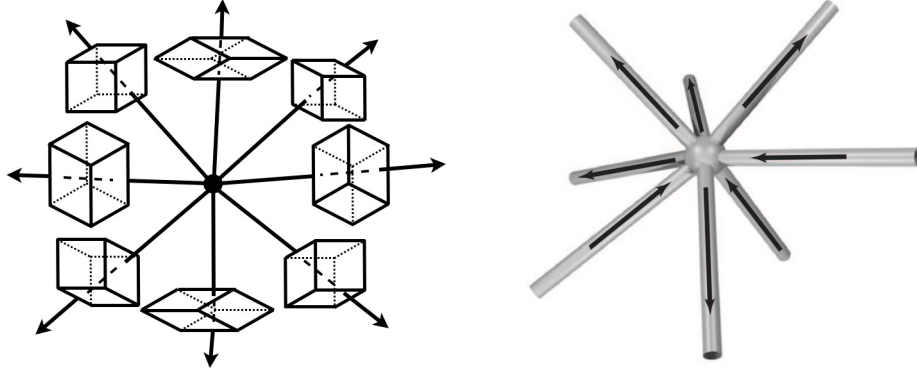
$$\sum_{\substack{L \supset V \\ \delta \circ \delta \equiv 0}} \sum_{h \supset L} \frac{1}{2} \mathbf{L} \cot(\theta_{Lh}) \epsilon_h + \mathcal{O}(L^5) = 0. \quad (14)$$

This is the RC formulation of the contracted Bianchi identity. The first term in this expression scales with  $\mathcal{O}(L^4)$ , since the deficit angles  $\epsilon_h$  scale as  $\mathcal{O}(L^2)$ . The contracted Bianchi identity is not identically zero because small, finite, rotations do not necessarily commute. Consequently, the final term scales with both the edge length  $L$  and the rotation commutator  $[\epsilon_h, \epsilon_{h'}]$ , yielding an overall  $\mathcal{O}(L^5)$  behaviour in the error term.

Two features are apparent in the discrete contracted Bianchi identity. First, it has the form of a Kirchhoff-like conservation law. The analysis presented in this manuscript completes the derivation of this Kirchhoff-like property of the contracted Bianchi identity. We have reduced the results of previous calculations from a non-local, boundary-valued sum to a vertex-based conservation equation. This was accomplished by utilizing the freedom we have in choosing the fulcrum for each of the moment of rotation trivectors, here we have chosen the vertex  $V$  common to all of the faces of the dual polytope  $V^*$ . Equivalently, this can be understood by our ability to parallel transport the moment of rotation trivector from the midpoint  $\mathcal{O}_i$  the edge  $L_i$  to the vertex  $V$ . No higher order corrections than corrections already discussed in this manuscript are introduced. It is this understanding of the relation between the contracted Bianchi identity gives rise to conservation of source which is vital to understanding how source can be coupled to the lattice. Second, the appearance of a 4-vector identity at each vertex signals that there are exactly four “approximate” diffeomorphic degrees of freedom per vertex in the simplicial lattice. This last point has been important for understanding the dynamical degrees of freedom in RC [15, 16, 17], and the resulting approximate diffeomorphism freedom has been utilized to solve the initial value problem [34].

#### 4. Expanding Regge Calculus Beyond the Vacuum

RC has been a predominately vacuum theory of gravitation with only specialized applications to spacetimes with non-gravitational sources [3]. Our approach can provide a consistent and generic coupling of matter to the simplicial lattice. The cornerstone of our approach relies on the nature of conservation principles of the lattice. We know of no better way to understand the conservation than through the BBP. This application of the BBP has led to an approximate conservation of moment-of-rotation for RC.



**Figure 4.** *The Kirchhoff-like form of the contracted Bianchi identity in RC:* On the left is an exploded view of the edges meeting a vertex  $V$  and their dual 3-volumes,  $L^*$ . The first step towards the Kirchhoff-like conservation principle is constructing the total moment of rotation for each of the dual 3-volumes,  $L^*$ . On the right is a depiction of the flow of moment-of-rotation, or equivalently flow of stress-energy along each of the edges,  $L$  meeting at the vertex  $V$ . The flow of moment-of-rotation entering or leaving vertex,  $V$ , is conserved to second order in the lattice spacing,  $L$ . Since the Einstein equations, and their RC equivalent, equates the moment-of-rotation with the stress-energy, this contracted Bianchi identity can be viewed as a circuit-like conservation law. Here the “wires” of the circuit are the edges of the simplicial lattice, and the “current” in each of the wires is the doubly-projected stress-energy tensor  $T_{LL}$  along the given edge  $L$  emanating from vertex  $V$ .

The conservation of moment-of-rotation takes a form which is ideally suited for applications to matter: the contracted Bianchi identity in RC becomes a Kirchhoff-like conservation principle along the edges emanating from a specific vertex. With this and Einstein’s equations, we obtain an approximate conservation equation for energy-momentum on the lattice,

$$\sum_{L \supset V} G_{LL} L^* = \sum_{L \supset V} \kappa T_{LL} L^* \cong 0. \quad (15)$$

In particular, the doubly projected stress-energy along the edges emanating from a vertex must sum to zero, to at least second order in the length scale of the lattice. While not exact, this gives the interpretation of a Kirchhoff-like conservation principle for the geometry, and (with Einstein’s equations) the flow of energy and momentum (Figure 4). As a result, we obtain a set of vertex-based constraints for edge-based expressions that constrain energy-momentum. This exercise indicates that energy-momentum is naturally wired to the simplicial lattice at each vertex and is naturally wired to each edge in its coupling with the simplicial field equations.

For applications of RC to pre-geometric quantum spacetime, one must necessarily

formulate an appropriate stress-energy tensor arising from the quantum dynamics. For applications to classical spacetimes a simplicial form of the stress-energy tensor must be constructed from the non-gravitational sources. This work indicates that the stress-energy will most naturally be expressed as a vertex-based tensor, and that its coupling to the RC equations will be through its double projection on the edges of the lattice. We will explore this coupling in future work.

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